

## MANIFOLDS OF ALMOST NONNEGATIVE RICCI CURVATURE

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To investigate relations between Ricci curvature and topology of a Riemannian manifold is one of the main subjects in differential geometry. In this paper, we study the global structure of manifolds of almost nonnegative Ricci curvature by means of the convergence and collapsing phenomena of Riemannian manifolds.

Let  $M$  be a compact connected  $C^\infty$  Riemannian manifold of dimension  $n$ . A classical theorem of Bochner states that if the Ricci curvature  $\text{Ric}(M)$  of  $M$  is nonnegative, then the first Betti number  $b_1(M)$  of  $M$  satisfies  $b_1(M) \leq n$ , where the equality takes place if and only if  $M$  is isometric to a flat torus. In [14], Gromov extended this result as follows: There is an  $\varepsilon > 0$  depending on  $n$  and a given constant  $D > 0$  such that if the diameter  $d(M)$  and the Ricci curvature of  $M$  satisfy  $d(M) \leq D$  and  $\text{Ric}(M) > -\varepsilon$ , then the first Betti number of  $M$  is still bounded by  $n$ . Gallot [10] also gave an analytic proof.

Under an additional condition, an upper bound of sectional curvature, we determine the topological and global geometric structure. Let  $\mathcal{M}(n, D)$  denote the family of compact Riemannian  $n$ -manifolds  $M$  with sectional curvature  $K(M) \leq 1$  and diameter  $d(M) \leq D$ . Our first result is a topological classification by first Betti numbers.

**Theorem 1.** *There is an  $\varepsilon > 0$  depending on  $n$  and  $D$  such that if  $M \in \mathcal{M}(n, D)$  satisfy  $\text{Ric}(M) > -\varepsilon$ , then  $M$  is a fiber bundle over a  $b_1(M)$ -torus. In particular, if  $b_1(M) = n - 1$ , then  $M$  is diffeomorphic to an infranilmanifold, and if  $b_1(M) = n$ , then  $M$  is diffeomorphic to an  $n$ -torus.*

A manifold  $M$  is called an *infranilmanifold* if a finite covering space of  $M$  is a quotient of a simply connected nilpotent Lie group by its lattice. The special case  $b_1(M) = n$  of Theorem 1 gives a partial affirmative answer to a conjecture of Gromov [14] (see the end of §2).

Our next goal is to describe the global geometric structure. To do this, we shall collapse Riemannian manifolds in the situation that the greatest lower bound of Ricci curvature tends to zero. In general cases, limit spaces

may have complicated singularities. In our situation, however, we obtain a stability theorem in the cases of large first Betti number.

**Theorem 2.** *Let  $\{M_i\}_{i \in \mathbb{N}}$  be a convergent sequence in  $\mathcal{M}(n, D)$  for the Hausdorff distance such that  $\text{Ric}(M_i) > -1/i$ ,  $b_1(M_i) \geq n-1$ . Then the limit space  $X$  is isometric to the quotient space of a flat torus by an isometric (not necessarily free)  $Z_2$  action. In the case when each  $M_i$  is orientable,  $X$  itself is a flat torus.*

As a result of Theorem 2, we obtain from Gromov's compactness theorem [14] that if  $M_i$  does not collapse, then the orientable double covering space  $\hat{M}_i$  of  $M_i$  converges to a flat  $n$ -torus for the Lipschitz distance, and from the fiber bundle theorem of Fukaya [9] that if  $M_i$  collapses, then there exists an almost Riemannian submersion from  $\hat{M}_i$  to a flat torus with an infranilmanifold as a fiber.

The main method in the proof which we employ is the  $C^{1,\alpha}$  convergence of Riemannian metrics with respect to harmonic coordinates developed by Greene and Wu [12] and Peters [23]. We shall recall it in §1 as preliminaries. The convergence method, in particular, in noncollapsing situations, has been used by some authors to obtain results about pinching problems ([2], [7], [8], [18], [19] etc.). We have to consider the collapsing situation, however. The key to the proof is to show the convergence of harmonic 1-forms. This is done in §2 with the proof of Theorem 1. The proof of Theorem 2 is deferred to §3.

The author would like to thank Professor Hajime Sato for his kind advice.

## 1. Preliminaries

In [14], Gromov introduced the notion of *Hausdorff distance* between metric spaces. For subsets  $A$  and  $B$  in a metric space  $Z$ , we denote by  $d_H^Z(A, B)$  the infimum of positive  $\varepsilon$  such that the  $\varepsilon$ -neighborhoods of  $A$  and  $B$  include  $B$  and  $A$  respectively. The Hausdorff distance between two metric spaces  $X$  and  $Y$  is, by definition

$$d_H(X, Y) = \inf_{Z, f, g} d_H^Z(f(X), g(Y)),$$

where  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$  are isometric imbeddings.

The *Lipschitz distance* between  $X$  and  $Y$  is defined as the infimum of numbers  $L$  such that

$$e^{-L} \leq d(f(x_1), f(x_2))/d(x_1, x_2) \leq e^L$$

for all  $x_1 \neq x_2$  in  $X$ , where  $f$  runs over the set of all bi-Lipschitz homeomorphisms of  $X$  onto  $Y$ . The convergence for the Hausdorff distance is characterized in terms of the Lipschitz convergence of discrete subsets: A

subset  $N$  in  $X$  is called an  $\varepsilon$ -net if and only if for any  $x$  in  $X$  there is  $x'$  in  $N$  such that  $d(x, x') < \varepsilon$ .

**Proposition** [14, 3.5]. (a) *If a sequence  $\{X_i\}$  of metric spaces converges to  $X$  for the Hausdorff distance, then for every positive  $\varepsilon$  and  $\varepsilon' > \varepsilon$ , every  $\varepsilon$ -net of  $X$  is the limit for the Lipschitz distance of a sequence  $N_i$  where  $N_i$  is an  $\varepsilon'$ -net of  $X_i$ .*

(b) *Conversely, if  $X$  and  $X_i$  have uniformly bounded diameters, and if for every  $\varepsilon > 0$  there exists an  $\varepsilon$ -net of  $X$  which is the limit for the Lipschitz distance of a sequence of  $\varepsilon$ -nets  $N_i$  of  $X_i$ , then  $X_i$  converges to  $X$  for the Hausdorff distance.*

Here we exhibit only a few examples which imply that the assumptions in Theorem 2 are optimal.

**Example 1.1** (*Klein bottle*). For each positive  $\varepsilon$ , consider a Klein bottle  $M_\varepsilon$  defined by the distance subgroup of isometries of the flat plane  $R^2$  with the generators  $\gamma_1, \gamma_2$ :

$$\gamma_1(x, y) = (x + 1, y), \quad \gamma_2(x, y) = (-x, y + \varepsilon).$$

Then  $M_\varepsilon$  converges for the Hausdorff distance to the interval  $S^1/Z_2 = [0, 1/2]$  as  $\varepsilon \rightarrow 0$ .

**Example 1.2** (*Three-dimensional nilmanifold*). Let  $N$  be the Heisenberg group and  $\Gamma$  the integer lattice:

$$N = \left\{ \begin{pmatrix} 1 & z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}; x, y, z \in R \right\}, \quad \Gamma = \left\{ \begin{pmatrix} 1 & c & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}; a, b, c \in Z \right\}.$$

Then the quotient space  $M = \Gamma \backslash N$  is orientable, and  $b_1(M) = 2$ . For every  $\varepsilon > 0$ , define the left invariant metric  $g_\varepsilon$  on  $M$  by

$$\begin{pmatrix} 0 & w & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \in T_e N, \quad \left\| \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\| = \varepsilon^2 u^2 + v^2 + w^2,$$

where  $e$  is the identity element of  $N$ . The sectional curvature and diameter of  $g_\varepsilon$  satisfy  $|K(g_\varepsilon)| \leq 24\varepsilon^2$ ,  $d(g_\varepsilon) \leq 2$  and  $(M, g_\varepsilon)$  converges to a flat torus  $T^2$  with respect to the Hausdorff distance.

Now we briefly sketch the  $C^{1,\alpha}$  convergence of Riemannian metrics developed by Greene and Wu [12] and Peters [23] (see also [14] and Katsuda [17]).

A coordinate system  $(h^1, \dots, h^n)$  defined on an open subset of a Riemannian manifold  $(M, g)$  is called a *harmonic coordinate system* if and only if  $\Delta h^i \equiv 0$ ,  $i = 1, \dots, n$ , where  $\Delta$  is the Laplacian of  $(M, g)$ . For positive constants  $\Lambda, D, \varepsilon$ , let  $\mathcal{C} = \mathcal{C}(n, \Lambda, D, \varepsilon)$  be the family of Riemannian  $n$ -manifolds

$M$  such that  $|K(M)| \leq \Lambda^2$ ,  $d(M) \leq D$ ,  $i(M) \geq \varepsilon$ . Let  $\alpha$ ,  $0 < \alpha < 1$ , be a fixed constant. By Jost and Karcher [16], there exists positive constants  $R' = R'(n, \Lambda, \varepsilon)$  and  $C = C(n, \Lambda, \varepsilon, \alpha)$  such that for each  $(M, g)$  in  $\mathcal{C}$  and  $x$  in  $M$ , there is a harmonic coordinate system  $\{h^i\}$  on the metric ball  $B_{R'}(x, M)$  in  $M$  around  $x$  of radius  $R'$  such that

$$|g_{ij}|_{C^{1,\alpha}}, |g^{ij}|_{C^{1,\alpha}} < C,$$

where  $g_{ij} = g(\partial/\partial h^i, \partial/\partial h^j)$  and the  $(1, \alpha)$  norms are taken in the  $h^i$  coordinates. Furthermore, the eigenvalues of  $(g_{ij})$  are uniformly bounded from below and from above by positive constants. Let  $H_r(z, M)$  be a harmonic  $r$ -ball around  $z$ , namely,

$$H_r(z, M) = \left\{ y \in B_{R'}(x, M); \sum_{i=1}^n (h^i(y))^2 \leq r \right\}$$

for harmonic coordinates  $h^i$  on  $B_{R'}(x, M)$ .

Let  $(M_k, g_k)$  be a convergent sequence in the family  $\mathcal{C}$  for the Hausdorff distance. Then the limit space  $X$  carries a  $C^{1,\alpha}$  Riemannian metric in the following way: There exist positive numbers  $R < R'$  and  $Q' \in \mathbb{Z}_+$  depending on  $n, \Lambda, D, \varepsilon$  such that each  $(M_k, g_k)$  in  $\mathcal{C}$  is covered by  $Q(k)$  harmonic  $R$ -balls  $\{H_R(z_i, M_i)\}_{i=1, \dots, Q(k)}$  with  $Q(k) \leq Q'$ . By passing to a subsequence, we may assume  $Q(k) \equiv Q$  for all  $k$ . Further, by passing to a subsequence and by using a center of mass technique, we obtain, for a fixed large  $i$  and every  $k \geq i$ , a diffeomorphism  $F = F_k: M \rightarrow M_k$ ,  $M := M_i$ , which is  $C^{2,\alpha}$  uniformly bounded in harmonic coordinates. Then for each  $H_R(z_l, M)$ ,  $1 \leq l \leq Q$ , with harmonic coordinates  $\{h^p\}$ ,

$$F^* g_k = g_{(k)ab} (\partial F^a / \partial h^p) (\partial F^b / \partial h^q) dh^p dh^q,$$

where  $F^a = \tilde{h}^a \circ F$ ,  $\{\tilde{h}^a\}$  being harmonic coordinates in  $\tilde{M} = M_k$ , and  $g_{(k)ab}$  is the components of  $g_k$  with respect to  $\{\tilde{h}^a\}$ . Each of  $g_{(k)ab}$ ,  $\partial F^a / \partial h^p$ ,  $\partial F^b / \partial h^q$  is uniformly bounded in  $C^{1,\alpha}$  norm. Hence by passing to a subsequence  $Q$  times,  $F^* g_k$  converges to a  $C^{1,\alpha}$  metric. We note that  $\partial(F^{-1})^p / \partial \tilde{h}^a$  is also uniformly bounded in  $C^{1,\alpha}$  norm with respect to  $\tilde{h}^a$  coordinates, where  $(F^{-1})^p = h^p \circ F^{-1}$ .

## 2. Convergence of harmonic 1-forms

In the nonnegative Ricci curvature case, every harmonic 1-form is parallel [3]. Together with this, the Hodge theory yields Bochner's result in the introduction. We look at the Bochner technique again from the viewpoint of convergence of Riemannian metrics.

Let  $(M_i, g_i)$  be a sequence in the family  $\mathcal{M}(n, D)$  such that  $b_1(M_i) \neq 0$  and  $\text{Ric}(M_i) > -\Lambda^2$  for a constant  $\Lambda > 0$ . Notice that the absolute value of the sectional curvature of  $M_i$  is uniformly bounded. Let  $p_i$  be a fixed point of  $M_i$  and let  $B$  and  $B'$  be the metric balls in  $R^n$  around the origin of radii 1 and 2 respectively. Let  $f_i: B' \rightarrow M_i$  be the composition of a linear isometry of  $B'$  onto  $B_2(0, T_{p_i}M_i)$  and the exponential mapping of  $M_i$  at  $p_i$ . By the curvature assumption,  $f_i$  has maximal rank, let  $\tilde{g}_i$  denote the metric of  $B'$  induced from  $f_i$ . Since the "injectivity radius" of  $(B, \tilde{g}_i)$  in  $(B', \tilde{g}_i)$  is uniformly bounded below by a positive constant, we can apply the  $C^{1,\alpha}$  convergence as in §1 to the sequence  $(B, \tilde{g}_i)$ : Cover  $(B, \tilde{g}_i)$  by harmonic balls in  $(B', \tilde{g}_i)$  whose centers lie in  $B$ . By passing to a subsequence, we obtain, for a fixed large  $i$  and every  $k \geq i$ , a diffeomorphism  $F = F_k$  of  $(B, \tilde{g}_i)$  into  $(B', \tilde{g}_k)$  which is  $C^{2,\alpha}$  uniformly bounded in harmonic coordinates. Then on each preferred harmonic ball in  $(B', \tilde{g}_i)$  with coordinates  $\{h^p\}$ ,

$$\hat{g}_k = \tilde{g}_{(k)ab}(\partial F^a / \partial h^p)(\partial F^b / \partial h^q) dh^p dh^q,$$

where  $\hat{g}_k = F^* \tilde{g}_k$ ,  $F^a = \tilde{h}^a \circ F$ ,  $\{\tilde{h}^a\}$  being harmonic coordinates with respect to  $\tilde{g}_k$ . Passing to a subsequence,  $\hat{g}_k$  converges to a  $C^{1,\alpha}$  metric  $\hat{g}$ .

**Notation.** For 1-forms  $\omega, \eta$  on a Riemannian manifold  $M$ , we define its inner product  $\langle \omega, \eta \rangle$  by

$$\langle \omega, \eta \rangle = \text{vol}(M)^{-1} \langle \omega, \eta \rangle_{L^2} = \text{vol}(M)^{-1} \int_M (\omega, \eta) dv,$$

where  $dv$  is the canonical measure of  $M$ .

Let  $\omega^k$  be a harmonic 1-form of  $(M_k, g_k)$  such that  $\langle \omega, \eta \rangle = 1$ , and let  $\hat{\omega}^k$  be its lift to  $B$  by  $G_k$ , where we set  $G_k = f_k \circ F_k$ .

**Proposition 2.1.** *A subsequence of  $\hat{\omega}^k$  converges to a  $C^{1,\alpha}$  1-form on  $B$ .*

*Proof.* By the Poincare Lemma, there exists a function  $f^k$  such that  $df^k = \hat{\omega}^k$ . Then  $\Delta f^k = \delta \hat{\omega}^k = 0$ , where  $\delta$  is the coboundary operator of  $(B, \hat{g}^k)$ . Namely, on each preferred harmonic ball in  $(B', \hat{g}_i)$  with coordinates  $\{h^p\}$ ,  $f^k$  satisfies

$$\hat{g}_{(k)}^{pq} (\partial^2 f^k / \partial h^p \partial h^q) + (\partial \hat{g}_{(k)}^{pq} / \partial h^p + \frac{1}{2} \hat{g}_{(k)}^{pq} \partial(\log \hat{G}_k) / \partial h^p) \partial f^k / \partial h^q = 0,$$

where  $\hat{g}_{(k)}^{pq}$  is the inverse of  $\hat{g}_{(k)pq}$ ,  $\hat{g}_{(k)pq} = \hat{g}_k(\partial / \partial h^p, \partial / \partial h^q)$ , and  $\hat{G}_k = \det(\hat{g}_{(k)pq})$ . From  $\hat{g}_{(k)pq} = \tilde{g}_{(k)ab}(\partial F^a / \partial h^p)(\partial F^b / \partial h^q)$ , we obtain uniform  $C^{0,\alpha}$  bounds of  $\partial \hat{g}_{(k)}^{pq} / \partial h^p$ ,  $\partial(\log \hat{G}_k) / \partial h^p$  and a uniform positive bound from below for the minimal eigenvalue of  $(\hat{g}_{(k)}^{pq})$ . It follows from the classical Schauder interior estimate that

$$|f^k|_{C^{2,\alpha}} \leq c_1 |f^k|_{C^0},$$

where and hereafter, we denote by  $c_i$  some uniform constants. By adding a constant to  $f_k$ , we may assume that  $|f^k|_{C^0} \leq c_2|\hat{\omega}^k|_{C^0} \leq c_2|\omega^k|_{C^0}$ . According to Li [20, Lemma 8] and Gallot [10, Proposition 3.2], we have

$$|\omega^k|_{C^0} \leq c_3\langle \omega^k, \omega^k \rangle^{1/2}.$$

It follows  $|f^k|_{C^{2,\alpha}} \leq c_1c_2c_3 = c_4$ , and hence  $|\hat{\omega}_p^k|_{C^{1,\alpha}} \leq c_4$  for  $\hat{\omega}^k = \hat{\omega}_p^k dh^p$ . Thus, by Ascoli's theorem, a subsequence of  $\hat{\omega}^k$  converges to a  $C^{1,\alpha}$  1-form.

Now we further assume for the sequence  $(M_i, g_i)$  that the Ricci curvature of  $M_i$  is greater than  $-1/i$ .

**Proposition 2.2.** *The limit 1-form is parallel with respect to the limit metric.*

*Proof.* Let  $\hat{\omega}$  be the limit  $C^{1,\alpha}$  1-form of  $\hat{\omega}^k$ . By the Weitzenböck formula,

$$-\frac{1}{2}\Delta(|\omega^k|^2) = |D^k\omega^k|^2 + \text{Ric}(M_k)(\sharp\omega^k, \sharp\omega^k),$$

where  $D^k$  is the covariant derivative with respect to  $g_k$  and  $\sharp\omega^k$  is the vector corresponding to  $\omega^k$ . Integrating this, we have

$$0 = \text{vol}(M_k)^{-1} \int_{M_k} |D^k\omega^k|^2 dv + \text{vol}(M_k)^{-1} \int_{M_k} \text{Ric}(M_k)(\sharp\omega^k, \sharp\omega^k) dv,$$

where

$$\liminf_{k \rightarrow \infty} \text{vol}(M_k)^{-1} \int_{M_k} \text{Ric}(M_k)(\sharp\omega^k, \sharp\omega^k) dv = 0,$$

from the assumption for Ricci curvature. Hence, by passing to a subsequence, we may assume

$$(*) \quad \lim_{k \rightarrow \infty} \text{vol}(M_k)^{-1} \int_{M_k} |D^k\omega^k|^2 dv = 0.$$

Now suppose that  $\hat{\omega}$  is not parallel with respect to the limit metric  $\hat{g}$ . We set  $\hat{A}_k = |\hat{D}^k\hat{\omega}^k|^2$ ,  $\hat{A} = |\hat{D}\hat{\omega}|^2$ , where  $\hat{D}^k$  and  $\hat{D}$  denote the covariant derivative with respect to  $\hat{g}_k$  and  $\hat{g}$ . Then there is a point  $m$  in  $B$  such that  $\hat{A} > a$  on  $B_\delta(m, (B, \hat{g}))$  for positive constants  $a$  and  $\delta$ . It turns out

$$(**) \quad \hat{A}_k > a/2 \quad \text{on } B_\delta(m, (B, \hat{g}_k))$$

for large  $k$ . We set  $A_k = |D^k\omega^k|^2$ . Together with this, the Bishop and Gromov volume comparison theorem [14] implies

$$1 - \text{vol}\{A_k \leq a/2\} / \text{vol}(M_k) \geq \text{vol } B_\delta(q_k, M_k) / \text{vol}(M_k) \geq b(\delta)/b(D),$$

where  $q_k = G_k(m)$  and  $b(\delta)$  denotes the volume of the  $\delta$ -ball in the simply connected space of constant curvature  $-1$ . Therefore,

$$\text{vol}\{A_k < a/2\} / \text{vol}(M_k) \leq 1 - b(\delta)/b(D)$$

for all large  $k$ . However, (\*) yields, for every fixed  $b > 0$ ,

$$\lim_{k \rightarrow \infty} \text{vol}\{A_k < b\} / \text{vol}(M_k) = 1.$$

This is a contradiction.

*Proof of Theorem 1.* We suppose the contrary. Then there exists a sequence  $\{M_k\}$  in  $M(n, D)$  such that  $\text{Ric}(M_k) > -1/k$ , and that each  $M_k$  does not admit any fibering over a  $b_1(M_k)$ -torus. We may assume  $b_1(M_k) \equiv s \neq 0$  for all  $k$ . Let  $\{\omega_i^k\}_{1 \leq i \leq s}$  be harmonic 1-forms of  $M_k$  giving a basis of the de Rham cohomology group such that  $\langle \omega_i^k, \omega_j^k \rangle = \delta_{ij}$ ,  $1 \leq i, j \leq s$ . Cover  $M_k$  by metric balls  $\{B_{k,\beta}\}_{1 \leq \beta \leq N(k)}$  of radius 1, where we may assume  $N(k) \equiv N$  for all  $k$ . For each fixed  $i$ ,  $1 \leq i \leq s$ , and  $\beta$ ,  $1 \leq \beta \leq N$ , let  $\hat{\omega}_{i,\beta}^k$  be the lift of  $\omega_i^k|_{B_{k,\beta}}$  to  $B$  as in the previous construction. Applying the previous argument to  $\{\hat{\omega}_{i,\beta}^k\}_k$  we obtain a parallel  $C^{1,\alpha}$  1-form  $\hat{\omega}_{i,\beta}$  on  $B$  with respect to a  $C^{1,\alpha}$  Riemannian metric. It is easily seen that the constant  $c_{ij} = \langle \hat{\omega}_{i,\beta}, \hat{\omega}_{j,\beta} \rangle$  does not depend on  $\beta$ . It follows from the following calculation that  $c_{ij} = \delta_{ij}$ :

$$\begin{aligned} |\delta_{ij} - c_{ij}| &\leq \lim_{k \rightarrow \infty} \text{vol}(M_k)^{-1} \int_{M_k} |(\omega_i^k, \omega_j^k) - c_{ij}| \, dv \\ &\leq \lim_{k \rightarrow \infty} \sup_{M_k} |(\omega_i^k, \omega_j^k) - c_{ij}| \\ &= \lim_{k \rightarrow \infty} \sup_B |(\hat{\omega}_{i,\beta}^k, \hat{\omega}_{j,\beta}^k) - c_{ij}| = 0. \end{aligned}$$

It turns out that  $\{\omega_i^k\}_{1 \leq i \leq s}$  are pointwise linearly independent for large  $k$ . By an argument of Tischler [25], (also compare Lichnerowicz [21]), we can find differentiable functions  $\varphi_i: M_k \rightarrow S^1$ ,  $1 \leq i \leq s$ , such that  $\varphi_i^*(dt) = \omega_i^k$ . The mapping  $\varphi = (\varphi_1, \dots, \varphi_s)$  is a submersion of  $M_k$  onto an  $s$ -torus  $T^s$ , and  $M_k$  is a fiber bundle over  $T^s$ .

It remains to show that an  $S^1$ -bundle  $M$  over  $T^{n-1}$  is diffeomorphic to an infranilmanifold. Let  $h$  be a flat metric on  $T^{n-1}$ . From the bundle structure, in a standard way, we can construct a metric  $g_\varepsilon$  on  $M$  for each  $\varepsilon > 0$  so that the projection  $(M, g_\varepsilon) \rightarrow (T^{n-1}, \varepsilon h)$  is a Riemannian submersion, the fibers are closed geodesic of length  $\varepsilon$  and that the sectional curvature of  $g_\varepsilon$  is uniformly bounded as  $\varepsilon \rightarrow 0$ . Since the diameter of  $g_\varepsilon$  goes to zero as  $\varepsilon \rightarrow 0$ , Gromov's almost flat manifolds theorem [13] implies that  $M$  is diffeomorphic to an infranilmanifold.

More generally, an  $S^1$ -bundle over an infranilmanifold is an infranilmanifold as the last part of the above proof shows.

In our proof of Theorem 1, of course, the condition for sectional curvature is essential. One might hope to eliminate it by using some methods different from ours, however. In fact, the most acceptable conjecture would be the following one (due to Gromov [14, 5.22] for the case  $b_1(M) = n$ ).

**Conjecture.** There is an  $\varepsilon > 0$  depending on  $n$  and  $D$  such that if  $M$  satisfies  $d(M) \leq D$  and  $\text{Ric}(M) > -\varepsilon$ , then  $M$  is a fiber bundle over a  $b_1(M)$ -torus.

### 3. Proof of Theorem 2

Let  $(M_k, g_k)$  be a sequence in  $\mathcal{M}(n, D)$  such that  $\text{Ric}(M_k) > -1/k$  and  $b_1(M_k) \geq n - 1$ , and assume that it converges to a metric space  $X$  with respect to the Hausdorff distance. For a given  $x$  in  $X$ , choose  $p_k$  in  $M_k$  such that  $\lim_{k \rightarrow \infty} d_H(B_r(p_k, M_k), B_r(x, X)) = 0$  for all  $r$ . Let  $B', B$  and  $G_k: B \rightarrow M_k$  be as before. The metric  $\hat{g}_k = G_k^* g_k$  converges to a  $C^{1,\alpha}$  metric  $\hat{g}$ .

**Lemma 3.1.**  $(B, \hat{g})$  is a  $C^\infty$  flat Riemannian manifold.

Let  $\omega_1^k, \dots, \omega_{n-1}^k$  be harmonic 1-forms on  $M_k$  such that  $\langle \omega_i^k, \omega_j^k \rangle = \delta_{ij}$ ,  $1 \leq i, j \leq n - 1$ . We set  $V_i^k = \sharp \omega_i^k$ . Let  $\hat{V}_i^k$  be the lift of  $V_i^k$  to the unit ball  $B$  by the map  $G_k$ , as in §2. By Proposition 2.2, for a subsequence,  $\hat{V}_i^k$  converges to a  $C^{1,\alpha}$  parallel vector field  $\hat{V}_i$  with respect to  $\hat{g}$ . Let  $\hat{V}_n$  be a unit vector field orthogonal to  $\{\hat{V}_1, \dots, \hat{V}_{n-1}\}$ , which is also parallel. Let  $\hat{\varphi}_{i,t}^k$  and  $\hat{\varphi}_{i,t}$ ,  $1 \leq i \leq n$ , be the flow transformations generated by  $\hat{V}_i^k$  and  $\hat{V}_i$ , where a priori  $\hat{\varphi}_{i,t}$  is only of class  $C^1$ . It is easily seen, however, that  $\hat{\varphi}_{i,t}^k$  gets close to an isometry as  $i \rightarrow \infty$ . Hence,  $\hat{\varphi}_{i,t}$  is an isometry, so has  $C^2$  differentiability by Calabi and Hartmann [5], which insures by a standard argument the existence of a coordinate system  $\{x^i\}$  on  $B$  such that  $\partial/\partial x^i = \hat{V}_i$ . Since  $\hat{g}(\partial/\partial x^i, \partial/\partial x^j) = \delta_{ij}$ ,  $\hat{g}$  is a flat metric.

The above argument together with Gromov's compactness theorem shows that if  $M_k$  does not collapse, then  $X$  is a flat  $n$ -manifold with  $b_1(X) \geq n - 1$ .

Now suppose that  $M_k$  collapses, namely, the injectivity radius of  $M_k$  goes to zero. We follow the argument in [14, 8.33–8.36]. For  $r = 1/5$ , let  $B_0 \subset B$  be the concentric ball of radius  $r$ , and let  $\Gamma_k$  be the pseudogroup of isometric imbeddings  $\gamma$  of  $(B_0, \hat{g}_k)$  into  $(B, \hat{g}_k)$  such that  $G_k \circ \gamma = G_k$ . Then  $B_r(p_k, M_k) = \Gamma_k \setminus B_0$ . Let  $Z$  be the set of imbeddings  $\varphi$  of  $B_0$  into  $B$  such that

$$\frac{1}{2} \leq d(\varphi(x), \varphi(y))/d(x, y) \leq 2,$$

where  $d$  is the distance induced from  $\hat{g}$ . By Ascoli's theorem,  $Z$  with the compact open topology is compact. By taking a subsequence, we may assume that  $\Gamma_k$  converges to a closed set  $\Gamma$  of  $Z$ , which is a pseudogroup of isometric imbeddings of  $(B_0, \hat{g})$  into  $(B, \hat{g})$ . Since  $\Gamma_k \setminus B_0$  converges to  $\Gamma \setminus B_0$  for the Hausdorff distance, it turns out that  $\Gamma \setminus B_0$  is isometric to  $B_r(x, X)$ .

Now assume that each  $M_k$  is orientable. Then we have



**Lemma 3.2.** *X is a flat manifold.*

*Proof.* Let  $V_n^k$  be the unit vector field orthogonal to  $\{V_1^k, \dots, V_{n-1}^k\}$  so that  $\{V_1^k, \dots, V_n^k\}$  gives the positive orientation, and let  $\hat{V}_n^k$  be the lift to  $B$ . For every  $u = (u^i)$  in  $R^n$ , the vector field  $\hat{V}^k = \sum_{i=1}^n u^i \hat{V}_i^k$  converges to a parallel field  $\hat{V}$ . For small  $t$ , let  $\hat{\varphi}_t^k$  and  $\hat{\varphi}_t$  be the flow transformations of  $\hat{V}^k$  and  $\hat{V}$  respectively. From the  $\Gamma_k$  equivariance of  $\hat{\varphi}_t^k$ , we obtain the  $\Gamma$  equivariance of  $\hat{\varphi}_t$ , namely,  $\gamma \circ \hat{\varphi}_t = \hat{\varphi}_t \circ \gamma$ ,  $\gamma \in \Gamma$ . It follows from the fact that  $\hat{\varphi}_t$  is a translation of the direction  $\sum u^i \hat{V}_i$ , that  $\Gamma$  consists of translations. Let  $\hat{\Gamma}$  be the extension of  $\Gamma$  as a closed subgroup of the vector group  $R^n$ , which is isomorphic to  $Z^k \times R^l$  for some  $k, l, k + l \leq n$ . Thus there is a natural imbedding of  $\Gamma \setminus B_0$  onto a neighborhood of the flat manifold  $\hat{\Gamma} \setminus R^n$ . Hence  $X$  admits the structure of a flat manifold.

To prove  $X$  is a flat torus, it suffices to show that  $X$  is homogeneous. Let  $x, y$  be any points in  $X$  with  $d(x, y) < 1/10$ . We have only to show the existence of an isometry carrying  $x$  to  $y$ . By the definition of the Hausdorff distance, we may assume that  $X$  and  $M_k$  are isometrically imbedded in a metric space  $(Z_k, d_k)$  so that  $d_H Z_k(X, M_k) < \varepsilon_k$ ,  $\lim \varepsilon_k = 0$ . Take  $x_k, y_k$  in  $M_k$  such that  $d_k(x_k, x) < \varepsilon_k$ ,  $d_k(y_k, y) < \varepsilon_k$ , and let  $\hat{x}_k, \hat{y}_k$  be points in  $B_0$  such that  $G_k(\hat{x}_k) = x_k$ ,  $G_k(\hat{y}_k) = y_k$ . By passing to a subsequence, we may assume that  $\hat{x}_k$  and  $\hat{y}_k$  converge to  $\hat{x}$  and  $\hat{y}$  respectively. Choose  $(u^i)$  in  $R^n$  and  $s$  so that the flow transformation  $\hat{\varphi}_t$  of  $\sum u^i \hat{V}_i$  satisfies  $\hat{\varphi}_s(\hat{x}) = \hat{y}$ . Then we have  $d(\hat{\varphi}_s^k(\hat{x}_k), \hat{y}_k) < \delta_k$ ,  $\lim \delta_k = 0$ , and hence  $d(\varphi_s^k(x_k), y_k) < \delta_k$ , where  $\hat{\varphi}_t^k$  and  $\varphi_t^k$  are the flow transformations of  $\sum u^i \hat{V}_i^k$  and  $\sum u^i V_i^k$  respectively. We set  $\varphi_k = \varphi_s^k$ . It follows from the lifting argument that  $\varphi_k$  is an almost isometry of  $M_k$ , that is,

$$|d(\varphi_k(p), \varphi_k(q)) - d(p, q)| < \mu_k$$

for all  $p, q$  in  $M_k$ , where  $\lim \mu_k = 0$ .

We next define a (not necessarily continuous) map  $\psi_k: X \rightarrow X$ . For any  $z$  in  $X$ , take  $z_k$  in  $M_k$  and  $z'$  in  $X$  such that  $d_k(z_k, z) < \varepsilon_k$ ,  $d_k(\varphi_k(z_k), z') < \varepsilon_k$ . Then we set  $\psi_k(z) = z'$ . The map  $\psi_k$  just defined satisfies

$$\begin{aligned} &|d(\psi_k(z), \psi_k(w)) - d(z, w)| \\ &< |d(\varphi_k(z_k), \varphi_k(w_k)) - d(z_k, w_k)| + 4\varepsilon_k < \mu_k + 4\varepsilon_k \end{aligned}$$

for every  $z, w$  in  $X$ , where  $w_k$  is a point in  $M_k$  with  $d_k(w_k, w) < \varepsilon_k$ , and

$$\begin{aligned} d(\psi_k(x), y) &\leq d_k(\psi_k(x), \varphi_k(x_k)) + d(\varphi_k(x_k), y_k) + d_k(y_k, y) \\ &< 2\varepsilon_k + \delta_k. \end{aligned}$$

Interpolating  $\psi_k$  by, for instance, a center of mass technique, we can construct a differentiable mapping  $\Phi_k: X \rightarrow X$  so as to satisfy the following:

$$\begin{aligned} |d(\Phi_k(z), \Phi_k(w)) - d(z, w)| &< \alpha_k, \\ d(\Phi_k(z), \psi_k(z)) &< \beta_k, \quad \|d\Phi_k\| \leq c, \end{aligned}$$

where  $\lim \alpha_k = 0$ ,  $\lim \beta_k = 0$  and  $c$  is a constant. By Ascoli's theorem, we may assume  $\Phi_k$  converges to an isometry  $\Phi$  of  $X$  satisfying  $\Phi(x) = y$ . Therefore,  $X$  is homogeneous, so a flat torus.

We should remark that, in the above proof, the diagonal method is also useful for the construction of  $\Phi$ .

Finally, we consider the nonorientable case. Suppose that each  $M_k$  is nonorientable, and take the orientable double covering space  $\hat{M}_k$  of  $M_k$ . By applying the previous argument to  $\hat{M}_k$ , and by passing to a subsequence if necessary, we may assume that  $\hat{M}_k$  converges to a flat torus  $T$  for the Hausdorff distance. Let  $\gamma_k$  be the deck transformation of  $\hat{M}_k$ . By the definition of Hausdorff distance, we may assume that  $\hat{M}_k$  and  $T$  are isometrically imbedded in a metric space  $(Z_k, d_k)$  in such a way that  $d_H Z_k(\hat{M}_k, T) < \varepsilon_k$ ,  $\lim \varepsilon_k = 0$ . By applying the previous argument to  $\gamma_k$  instead of  $\varphi_k$ , we can construct a differentiable mapping  $\Phi_k: T \rightarrow T$  so as to satisfy

- (i)  $d_k(\Phi_k(x), \gamma_k(x_k)) < \delta_k$  for  $x \in T$ ,  $x_k \in M_k$  with  $d_k(x, x_k) < \varepsilon_k$ ,
- (ii)  $|d(\Phi_k(x), \Phi_k(y)) - d(x, y)| < \mu_k$ ,
- (iii)  $d(\Phi_k^2(x), x) < \nu_k$ ,
- (iv)  $\|d\Phi_k\| \leq c$ ,

where  $\delta_k$ ,  $\mu_k$  and  $\nu_k$  tend to zero as  $k \rightarrow \infty$ . By Ascoli's theorem, we may assume that  $\Phi_k$  converges to an involutive isometry  $\Phi$  of  $T$ . It remains to prove that  $X$  is isometric to the quotient space  $T/\Phi$ . This is done in the following

**Assertion.**  $\lim_{k \rightarrow \infty} d_H(M_k, T/\Phi) = 0$ .

Let  $\pi: T \rightarrow T/\Phi$  and  $\pi_k: \hat{M}_k \rightarrow M_k$  be the projections, and let  $\varepsilon$  be a positive number. Let  $N_\varepsilon = \{y_j\}$  be a maximal set of points in  $T/\Phi$  such that any two distinct points have distance greater than  $\varepsilon$ . Set  $\pi^{-1}(y_j) = \{x_j, \Phi(x_j)\}$ , and take  $p_j$  in  $\hat{M}_k$  with  $d_k(p_j, x_j) < \varepsilon_k$ . Then  $N_\varepsilon^k = \{\pi_j(p_j)\}$  is a  $2\varepsilon$ -net in  $M_k$  for large  $k$ , and  $N_\varepsilon^k$  converges to  $N_\varepsilon$  for the Lipschitz distance. Hence  $M_k$  converges to  $T/\Phi$  for the Hausdorff distance.

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